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COMPACTLY GENERATED EXTENSIONS IN LOCALLY COMPACT ABELIAN GROUPS

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Abstract

In this paper, we define the set of all compactly generated extensions in the category of locally compact Abelian groups and prove that under the Baire sum, it forms an Abelian group. Also, we show that if G has only one open compact subgroup, say H, then the group of all compactly generated extensions of locally compact groups imbeds in the group of all continuous homomorphisms of G into H.

1. Introduction

Let $\mathcal{L}$ be the category of all locally compact Abelian groups with continuous homomorphisms as morphisms. An extension of G by A is a short exact sequence $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 0$, where i is a homeomorphism of A onto a closed subgroup of E and π is open. The set of equivalence classes of extensions of A by G is denoted by $Ext(G, A)$. A continuous section of an extension $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 0$ of A by G is a continuous map s such that $s : G \to E$ and $\pi s(x) = x$ for each

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$x \in G$. For example, if G is a connected locally compact group, then any extension of G by a connected simply connected Lie group has a continuous section [6].

It is known that [4], the class $Ext(G, A)$ of all congruence classes of extensions of A by G in $\mathcal{L}$ is an Abelian group with respect to the operation defined by

$$[E_1] + [E_2] = [\nabla_A (E_1 \oplus E_2) \Delta_G],$$

where E_1 and E_2 are extensions of A by G and Δ_G and ∇_A are the diagonal and codiagonal morphisms, respectively, that is,

$$\nabla_A : A \oplus A \to A, \qquad \nabla_A(a_1, a_2) = a_1 + a_2;$$

$$\Delta_G : G \to G \oplus G, \qquad \Delta_G(g) = (g, g).$$

A homomorphism is called *proper*, if it is open onto its image. If A and G are in $\mathcal{L}$, then an extensions of A by G is a proper short exact sequence $E : 0 \to A \xrightarrow{i} Q \xrightarrow{\pi} G \to 0$, where i and ϕ are morphisms in $\mathcal{L}$. We will denote it by (E, π).

In [5], we introduced the compactly generated short exact sequences in $\mathcal{L}$. Let $\mathcal{C}$ be the category of all groups in $\mathcal{L}$ that are compactly generated. The extension (E, π) in $\mathcal{L}$, is called *compactly generated,* if there exists an extension

$$0 \longrightarrow A \xrightarrow{i'} L \xrightarrow{\pi'} H \longrightarrow 0,$$

where L, H in $\mathcal{C}$, and there is a continuous homomorphism $\phi : Q \to L$ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i'"] & L \arrow[r, "\pi'"] & H \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[u, equal] & Q \arrow[r, "\pi"] \arrow[u, "\phi"] & G \arrow[r] \arrow[u, "j"] & 0.
\end{tikzcd}$$

For example, if G is locally compact, σ- compact and A is a compact closed subgroup of G, then the short exact sequence $0 \to \mathrm{A} \longrightarrow \mathrm{Q} \longrightarrow \mathrm{G} \to 0$ in $\mathcal{L}$ is a compactly generated sequence [5].

Two compactly generated extensions, $0 \to A \xrightarrow{i_1} Q \xrightarrow{\pi_1} G \to 0$ and $0 \to A \xrightarrow{i_2} Q' \xrightarrow{\pi_2} G \to 0$ are *equivalent* in $\mathcal{L}$, if and only if there exists a continuous homomorphism $\psi : Q \to Q'$ making the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_1} & Q & \xrightarrow{\pi_1} & G & \longrightarrow & 0 \\ & & \| & & \downarrow\psi & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{i_2} & Q' & \xrightarrow{\pi_2} & G & \longrightarrow & 0, \end{array}$$

where by [2, Corollary 2.2], ψ is an isomorphism in $\mathcal{L}$.

The set of all congruence classes of compactly generated extensions of A by G is denoted by $Ext_c(G, A)$.

In this paper, we will prove that $Ext_c(G, A)$ is an Abelian subgroup of $Ext(G, A)$ with respect to the Baire sum. Furthermore, let G be in $\mathcal{L}$ and A be a compactly generated G-module. We show that if G has only one open compact subgroup, say H, then $Ext_c(G, A)$ imbeds in $Hom(G, H)$, where the $Hom(G, H)$ is an Abelian topological group with respect to the compact-open topology under point-wise addition [4].

We recall that an Abelian group G is *divisible*, if $G = mG$ for every integers m. An Abelian group G is reduced, if the only divisible subgroup of G is the trivial one.

2. Some Results on $Ext_c(G, A)$

In this section, we show that $Ext_c(G, A)$ is an Abelian subgroup of $Ext(G, A)$. Also, we prove that $Ext_c(G, A)$ can be embedded in $Hom(G, H)$.

Lemma 2.1. *Let G be in $\mathcal{L}$ and A be a compactly generated G-module. Then, the extension $0 \to A \xrightarrow{i} Q \xrightarrow{\pi} G \to 0$ in $\mathcal{L}$ is a compactly generated extension.*

Proof. Since G is a locally compact group, it is of the form $R^n \times K$, where K contains an open compact subgroup, say H [3]. Also $A \times H$ is compactly generated. Then, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i'} & A \times H & \xrightarrow{p} & H & \longrightarrow & 0 \\ & & \| & & \uparrow\phi & & \uparrow\pi_1 & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & Q & \xrightarrow{\pi} & G & \longrightarrow & 0, \end{array}$$

where $p : A \times H \to H$ is the projection and $\pi_1 : R^n \times H \to H$ with $\pi_1(\mathbf{x}, h) = x$, for all $\mathbf{x} \in R^n$, $h \in H$, and i' is injection.

Let $g : H \to A \times H$ with $g(h) = (e, h)$ for all $h \in H$ and e is identity element in A, then we define continuous homomorphism $\phi = g(\pi_1(\pi))$.

We recall that for any two continuous homomorphisms $\phi_1 : A_1 \to B_1$ and $\phi_2 : A_2 \to B_2$, there is a continuous homomorphism

$$\phi_1 \oplus \phi_2 : A_1 \oplus A_2 \to B_1 \oplus B_2,$$

with the usual properties $(\phi_1 \oplus \phi_2)(\phi_1' \oplus \phi_2') = \phi_1\phi_1' \oplus \phi_2\phi_2'$ and $1_A \oplus 1_B = 1_{A\oplus B}$. So, this homomorphism may be defined by

$$(\phi_1 \oplus \phi_2)(a_1, a_2) = (\phi_1 a_1, \phi_2 a_2).$$

Given two compactly generated extensions:

$E_1 : 0 \to A_1 \xrightarrow{i_1} Q_1 \xrightarrow{\pi_1} G_1 \to 0$ and $E_2 : 0 \to A_2 \xrightarrow{i_2} Q_2 \xrightarrow{\pi_2} G_2 \to 0$, we show that their direct sum is a compactly generated extension. We have two commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i_1'} & L_1 & \xrightarrow{\pi_1'} & H_1 & \longrightarrow & 0 \\
 & & \| & & \uparrow\phi_1 & & \uparrow\psi_1 & & \\
0 & \longrightarrow & A_1 & \xrightarrow{i_1} & Q_1 & \xrightarrow{\pi_1} & G_1 & \longrightarrow & 0,
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \xrightarrow{i_2'} & L_2 & \xrightarrow{\pi_2'} & H_2 & \longrightarrow & 0 \\
 & & \| & & \uparrow\phi_2 & & \uparrow\psi_2 & & \\
0 & \longrightarrow & A_2 & \xrightarrow{i_2} & Q_2 & \xrightarrow{\pi_2} & G_2 & \longrightarrow & 0,
\end{array}$$

where H_1 and H_2 are open compact subgroups of G_1 and G_2, respectively. Then, we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \oplus A_2 & \xrightarrow{i_3'} & L_1 \oplus L_2 & \xrightarrow{\pi_3'} & H_1 \oplus H_2 & \longrightarrow & 0 \\
 & & \| & & \uparrow\phi_3 & & \uparrow\psi_3 & & \\
0 & \longrightarrow & A_1 \oplus A_2 & \xrightarrow{i_3} & Q_1 \oplus Q_2 & \xrightarrow{\pi_3} & G_1 \oplus G_2 & \longrightarrow & 0,
\end{array}$$

where $\phi_3 = \phi_1 \oplus \phi_2$, $\psi_3 = \psi_1 \oplus \psi_2$, $i_3 = i_1 \oplus i_2$, $\pi_3 = \pi_1 \oplus \pi_2$, $i_3' = i_1' \oplus i_2'$, and $\pi_3' = \pi_1' \oplus \pi_2'$. The above diagram is commutative, because

$$\begin{aligned}
\psi_3\pi_3 &= (\psi_1 \oplus \psi_2)(\pi_1 \oplus \pi_2) \\
&= \psi_1\pi_1 \oplus \psi_2\pi_2 = \pi_1'\phi_1 \oplus \pi_2'\phi_2 \\
&= (\pi_1' \oplus \pi_2')(\phi_1 \oplus \phi_2) \\
&= \pi_3'\phi_3.
\end{aligned}$$

Also, $H_1 \oplus H_2$ and $L_1 \oplus L_2$ are compactly generated, then $E_1 + E_2$ is a compactly generated extension.

The class of the split extension in $\mathcal{C}$ is the zero element of this group, while the inverse of any compactly generated extension E is the extension $(-1_A)E$, where $(-1_A)E$ is obtained from E just by changing the sign of the map $i : A \to Q$ and hence by changing the signs of the factor system $f, g : G \times G \to A$.

In fact, we have proved the following:

Theorem 2.2. *Let G be in $\mathcal{L}$ and A be a G-module. The set $Ext_c(G, A)$ of all congruence classes of compactly generated extensions of A by G is an Abelian group under the congruence classes of extensions E_1 and E_2 the congruence class of the extension*

$$E_1 + E_2 = [\nabla_A(E_1 \oplus E_2)\Delta_G].$$

The class of the split extension $0 \to A \to A \oplus G \to G \to 0$ in $\mathcal{C}$ is the zero element of this group, while the inverse of any E is the extension $(-1_A)E$.

Now, we can characterize the compactly generated extension groups. Let G be in $\mathcal{L}$ and A be a compactly generated G-module. Then G contains an open compact subgroup, say H [3].

Theorem 2.3. *Let G be in $\mathcal{L}$ and A be a compactly generated G-module. If G has only one non trivial open compact subgroup, say H, then $Ext_c(G, A)$ imbeds in $Hom(G, H)$.*

Proof. G contains an open compact subgroup, say H [3]. Let (E, π) be in $Ext_c(G, A)$, by definition, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i'} & L & \xrightarrow{\pi'} & H & \longrightarrow & 0 \\
 & & \| & & \uparrow \phi & & \uparrow j & & \\
0 & \longrightarrow & A & \xrightarrow{i} & Q & \xrightarrow{\pi} & G & \longrightarrow & 0,
\end{array}$$

consequently, for any element (E, π) in $Ext_c(G, A)$, there is an element of $Hom(G, H)$, say j, then we have

$$\zeta : Ext_c(G, A) \to Hom(G, H)$$

$$\zeta(E, \pi) = j.$$

Corollary 2.4. *Let G be in $\mathcal{L}$, A be a compactly generated G-module, and G has only one open compact subgroup, say H. If G is divisible and H is reduced, then $Ext_c(G, A) = 0$.*

Proof. By [1], $Hom(G, H)$ vanishes. So by Theorem 2.3, $Ext_c(G, A) = 0$.

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